Examples of Black Holes in Two-Time Physics

Sascha Vongehr
Department of Physics and Astronomy
University of Southern California
Los Angeles, CA 90089-0484
USA

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Abstract

Two time theory is derived via localization of the global Sp(2) symmetry [or Osp(1/2), Osp(N/2), Sp(2N),...] in phase space in order to give a self contained introduction to two time theory. Then it is shown that from the two-times physics point of view theories of point particles on many known black hole backgrounds are Sp(2) gauge duals of one another and of course also gauge dual to all other equal dimensional gauges from earlier two time related publications (hydrogen atom, ...). We reproduce the free (quantum) relativistic particle on 1+1 dimensional black hole backgrounds and 2+1 dimensional BTZ ones. Other 2+1 black holes and n+1 ones are touched on but explicitely found only as cross sections of complicated (n+1)+1 backgrounds. Further we give near horizon solutions (e.g. n+1 Robertson-Bertotti). Since two time physics can reproduce these backgrounds all particle actions have hidden symmetries that oftentimes have not been noticed before or whose origin was unclear.

1 Two Time Theory

1.1 Introduction

Early attempts of using two times go back as far as 1936 when Paul Dirac rederived $d = 3 + 1$ Maxwell theory from a manifestly conformally invariant $d = 4 + 2$ conformal space [1]. In order to remove the additional dimensions constraints had to be imposed artificially. We will derive two time theory from a completely different point of view. The need of two times follows rather being the starting point and the constraints are supplied by the theory itself.

The huge success of today's fundamental theories like the standard model and general relativity is due to localizations of global gauge freedoms. There is one global symmetry that has never been localized. It is the $Sp(2)$ symmetry in phase space. Just like one arrives at general relativity via localization of the Poincare symmetry, localizing the $Sp(2)$ symmetry in phase space tells us that we need two times. Here we will derive two time theory from this point of view with a formalism [2] that facilitates generalizations. We will show this for the introduction of spin. All that will be needed is to substitute the $Sp(2)$ group metric with the $Osp(1/2)$ group metric. For the $Sp(2)$ case we will derive here for the first time black hole backgrounds from two time theory. The derivation of two time theory and the finding of examples - here black holes - should serve as a self contained introduction into two time physics. We will restrict us to bosonic examples, i.e. consider only $Sp(2)$ duals. For the motivations of two time theory please see [2] where we show as well motivations due to considerations of M-theory. Only two time theory can give a large enough supergroup to embed M-theory, thus two time theory should lead to M-theory but there is only a toy-model [3] yet.

1.2 Derivation of Two Time Theory

1.2.1 The General Formalism

For a group with metric g and group elements J acting on a multiplet Φ according to $\Phi' = J\Phi$ (or $\Phi'_j = J^i_j \Phi_i$) the following is an invariant:

$$\Phi^{\dagger'} g \Phi' = \Phi^\dagger g \Phi \tag{1}$$

$\Phi^\dagger \neq \Phi$ since one is a row and the other one a column but it will turn out that $\Phi_i^\dagger = \pm\Phi_i$. For infinitesimal transformations we can write $J = E + g\omega$ where E is the unit matrix and ω is called the generator. If Φ depends on a parameter τ but J may not then this is a global symmetry. In order to make the symmetry local one introduces the covariant derivative $;_\tau = (,_\tau - gA)$ where A is the gauge field and g still the group metric (We do not consider a coupling strength.).

We choose the following for the group metric

$$g^\dagger = g^{-1} = -g \Rightarrow g^2 = -E \tag{2}$$

in order to ensure an anti-Hermitian norm and

$$\omega = \omega^\dagger \Rightarrow A^\dagger = A \tag{3}$$

We introduce the simplest (and well known) expressions that one could write down for an action and the Lagrangian in order to get the dimension of energy later on when Φ includes X and P:

$$S = \int_0^T \pounds d\tau \qquad \pounds = \frac{1}{2}\Phi^\dagger;_\tau g\Phi = \frac{1}{2}\Phi^\dagger,_\tau g\Phi - \frac{1}{2}\Phi^\dagger A\Phi \tag{4}$$

The Euler Lagrange equations give the equations of motion as:

$$\Phi,_\tau = gA\Phi \Rightarrow \pounds^\dagger = \pounds \tag{5}$$

$$\Phi_i\Phi_j = 0 \tag{6}$$

The latter tells us that we will only get trivial solutions if we do not write

$$\Phi_i\Phi_j = \Phi_i^M\Phi_j^N\eta_{MN} = 0 \tag{7}$$

where η has at least two times and is therefore the $SO(d,2)$ metric. For clarity we will see all this with an example:

1.2.2 $Sp(2)$ - Example

$Sp(2)$ satisfies the condition (2). The symmetry acts on the doublet

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} X \\ P \end{pmatrix} \tag{8}$$

where $(g_{ij}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $(g^{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The constraint (6) translates into

$$\Phi_i \Phi_j = 0 \Rightarrow X \cdot X = 0 = X \cdot P = 0 = P \cdot P \tag{9}$$

Thus it follows that X and P are two light like vectors that are not parallel. Therefore they must have two time like dimensions and they must have $d+2$ entries. We write X^M where $M \in \{0, 0', 1, ...\}$ and X^M is now a $SO(d,2)$ vector. Thus we see via the localization that nature insists on two times because this is the only way not to get a trivial theory after gauge fixing all three gauge choices that $Sp(2) \simeq Sl(2,R) \simeq SO(2,1)$ offers. Every gauge choice means one choice of functional dependence of one of the Φ^i_M in terms of the others or put to equal a constant for example. Note that the two times do not come from a special choice of Lagrangian. We just used the simplest Lagrangian that gives us an energy $X_{,\tau} P$. Taking the localisation of the $Sp(2)$ of ordinary physics seriously is leading to the two times.

The three gauge choices fix a gauge surface inside the $SO(d,2)$ symmetric starting space such that we are left with $SO(d-1,1)$. The third gauge choice fixes the parametrization of the world line $\tau_{(t)}$. Note that we must have one more time and one more spatial direction. We write X^M where $M \in \{0, 0', 1, 1', 2, 3, ..., d-2\}$. The "accident" $Sp(2) \simeq SO(2,1)$ means that we can interpret the above as conformal gravity on the world line. We rewrite the quantization $[X,P] = i\hbar$ as

$$\Phi_i g^{ij} \Phi_j = i\hbar \tag{10}$$

1.2.3 The General Formalism Again

η_{MN} could be $diag(-1,-1,1,...,1)$ or related. The global $SO(d,2)$ contains d-dimensional Poincare symmetry $ISO(d-1,1)$ but there is no translation invariance in $d+2$ dimensions. The $SO(d,2)$ generators are due to Noether's theorem

$$J^{MN} = \frac{1}{4} \Phi_i^{[M} g^{ij} \Phi_j^{N]} \tag{11}$$

For the Casimirs holds classically

$$C_{n[SO(d,2)]} = \frac{1}{n!} Tr\,(iJ)^n \propto \Phi^M \Phi_M = 0 \tag{12}$$

,that means that for all gauges one might choose later on the Casimir invariants specify a unique representation of $SO(d,2)$. It completely characterizes the gauge invariant physical space. The same holds for the quantized theory but with $C_{n[SO(d,2)]} \neq 0$.

1.2.4 $Osp(1/2)$ - and $Osp(N/2)$ - Example

$Osp(1/2)$ satisfies the condition (2), too. The symmetry acts on

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} = \begin{pmatrix} X \\ P \\ \Psi \end{pmatrix} \tag{13}$$

where $(g_{ij}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}$ and $(g^{ij}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}$. The constraint (6) translates into

$$\Phi_i \Phi_j = 0 \Rightarrow X \cdot X = X \cdot P = P \cdot P = X \cdot \Psi = P \cdot \Psi = 0 \tag{14}$$

We have now five constraints matching the five parameters of $Osp(1/2)$. We can interpret the above as conformal supergravity on the world line. Introducing another arbitrary multiplet Θ_i the scalar product $\Phi g \Theta$ should be Hermitian and with $\Phi^\dagger \neq \Phi$ (one is a row, the other one a column) and for the components

$$\Phi_i^\dagger = \pm \Phi_i \tag{15}$$

follows that Ψ is Grassmanian. Thus, the quantization $\Phi_i g^{ij} \Phi_j = i\hbar$ (10) leads to $[X, P] = i\hbar$ again. The $\Phi_i^\dagger = \pm\Phi_i$ in (15) can be shown to be $\Phi_i^\dagger = +\Phi_i$ (It follows from the properties of the generator ω).

One might like to generalize to $Osp(N/2)$ using Ψ_a^M . Straightforward calculation results in:

$$J^{MN} = X^M P^N - P^N X^M + \frac{1}{2i} \sum_a \left(\Psi_a^M \Psi_a^N - \Psi_a^N \Psi_a^M \right) \tag{16}$$

$$\pounds = \dot{X} \cdot P - \frac{i}{2} \sum_a \dot{\Psi}_a \, \Psi_a - \frac{1}{2} A^{ij} \Phi_i \Phi_j - \frac{1}{2} A^{ab} \Psi_a \Psi_b + A^{ai} \Phi_i \Psi_a \tag{17}$$

where $i \in \{1, 2\}$. The generators show a spin and an angular momentum part $J^{MN} = L^{MN} + S^{MN}$. The first order form for the Lagrangian has been derived via dropping of total derivatives. It holds

$$\pounds,_{\dot{X}} = P \tag{18}$$

$$\pounds,_{\dot{\Psi}} = -\frac{i}{2} \Psi \tag{19}$$

,the latter being a second class constraint.

1.2.5 The General Formalism Again

Starting with the general formalism one only needs to specify the group metric g in order to get different theories. We had $Sp(2)$ for a simple point particle, $Osp(N/2)$ for a spinning particle but we could use for example $Sp(2N)$ for a N - particle system.

As we will see with the help of examples, the described theory is a consistent and unitary quantum theory. The $SO(d,2)$ covariant quantization gives the Casimir which for the bosonic point particle theory for example is $C_{2[SO(d,2)]} = 4C_{2[Sp(2)]} - \frac{1}{4}(d^2-4)$ where $C_{2[Sp(2)]} = 0$ because the physical states are singlets in the gauge invariant sector. Thus we get

$$C_{2[SO(d,2)]} = -\frac{1}{4}(d^2-4) \tag{20}$$

and the canonical or field theoretical quantization after gauge fixing always gives $C_2 = 1 - \left(\frac{d}{2}\right)^2$ also [4]. This generalizes according to the group used. For instance $Osp(N/2)$ leads to

$$C_{2[SO(d,2)]} = \frac{1}{8}(d+2)\,(N-2)\,(d+N-2) \tag{21}$$

There have been discussions in the past [5] [6] [7] [8] of formalisms with two times, but not including the non-trivial classical and quantum solutions of two time physics [4]. [5] suggested an action which can be gotten from our first order formalism by integrating out the momenta P in the path integral or semi-clasically by using its equation of motion. That second order formalism was obtained with different reasoning and motivation [5] , and without the concept of gauge duality.

2 $1+1$ Dimensional Black Hole Gauges

The $1+1$ black hole is a solution of two-time theory in 2+2 dimensions by choosing the $Sp(2,R)$ gauges $X^{+'} = 1$, $P^{+'} = 0$ (the third gauge is not yet fixed) and solving the constraints $X^2 = X \cdot P = 0$ as follows

$$M \;=\; (+' \quad -' \quad + \quad -) \tag{22}$$

$$
\begin{aligned}
X^M &= (1, \quad -uv, \quad u, \quad v)\, N_{(u,v)} \\
P^M &= \frac{1}{N}\,(0, \quad up_u + vp_v, \quad -p_v, \quad -p_u) && (23) \\
dX^M &= (0, \quad -d\,(uv)\,, \quad du, \quad dv)\, N + \frac{X^M}{N} dN && (24)
\end{aligned}
$$

The metric is given by $\eta^{+'-'} = \eta^{+-} = -1$ and the line element is

$$
(ds)^2 = (dX^M)(dX_M) = -2N^2 dudv \tag{25}
$$

Inserting these forms in the original $\mathrm{Sp}(2, R)$ local and $\mathrm{SO}(2,2)$ global invariant Lagrangian [4] gives

$$
\begin{aligned}
\pounds &= \dot{X}\cdot P - \frac{1}{2}A^{22} P\cdot P - \frac{1}{2}A^{11} X\cdot X - A^{12} X\cdot P && (26) \\
&= \dot{u}p_u + \dot{v}p_v + \frac{A^{22}}{N^2} p_u p_v && (27) \\
&= \frac{-N^2}{A^{22}}\dot{u}\,\dot{v} = \frac{1}{2A^{22}} G_{\mu\nu}\dot{x}^\mu \dot{x}^\nu && (28)
\end{aligned}
$$

The metric is recognized in the line element or in the last line $G_{\mu\nu} = \eta_{\mu\nu} N^2$, which is obtained by integrating out the momenta. This Lagrangian describes a particle moving in the background of the black hole given a suitable N.

This form shows that the system has the larger symmetry $\mathrm{SO}(2,2)$ whose generators are the Lorentz generators in the 2+2 dimensional space $L^{MN} = X^M P^N - X^N P^M$. In the present gauge these take a form that is quantum ordered already:

$$
\begin{aligned}
L^{+'-'} &= up_u + vp_v, \quad L^{+'+} = -p_v && (29) \\
L^{+'-} &= -p_u, \quad L^{-'+} = -u^2 p_u && (30) \\
L^{-'-} &= -v^2 p_v, \quad L^{+-} = -up_u + vp_v && (31)
\end{aligned}
$$

Under the $\mathrm{SO}(2,2) = SL\,(2,R)_L \otimes SL\,(2,R)_R$ the generators may be reclassified in the form

$$
\begin{aligned}
G_2^L &= vp_v, \quad G_+^L = G_0^L + G_1^L = -p_v, \quad G_-^L = G_0^L - G_1^L = -v^2 p_v \ , && (32) \\
G_2^R &= up_u, \quad G_+^R = G_0^R + G_1^R = -p_u, \quad G_-^R = G_0^R - G_1^R = -u^2 p_u \ . && (33)
\end{aligned}
$$

where $G_0^{L,R}$ are the compact generators and $G_{1,2}^{L,R}$ are the non-compact ones.

The classical SO$(2,2)$ symmetry transformations generated by these are obtained by evaluating the Poisson brackets $\delta u = \frac{1}{2}\varepsilon_{MN}\left\{L^{MN},u\right\}$, $\delta v = \frac{1}{2}\varepsilon_{MN}\left\{L^{MN},v\right\}$, which give

$$\begin{aligned}\delta u &= -\varepsilon_{+'-'}u+\varepsilon_{+'+}0+\varepsilon_{+'-}+\varepsilon_{-'+}u^2+\varepsilon_{-'-}0+\varepsilon_{+-}u, &(34)\\ \delta v &= -\varepsilon_{+'-'}v+\varepsilon_{+'+}+\varepsilon_{+'-}0+\varepsilon_{-'+}0+\varepsilon_{-'-}v^2-\varepsilon_{+-}v. &(35)\end{aligned}$$

The Lagrangian density transforms as follows

$$\delta\left(\dot{u}\,\dot{v}N^2\right)=\dot{u}\,\dot{v}\Lambda\left(\varepsilon,u,v\right) \tag{36}$$

This transformation is cancelled by

$$\delta\left(\frac{1}{A^{22}}\right)=\frac{-\Lambda\left(\varepsilon,u,v\right)}{A^{22}} \tag{37}$$

So that the Lagrangian is invariant under SO$(2,2)$

$$\delta\pounds=0. \tag{38}$$

This larger symmetry of the particle action in for example the SL$(2,R)/R$ black hole was not noticed before.

2.1 Example: The SL$(2,R)/R$ Black Hole Gauge

The SL$(2,R)/R$ black hole is described in terms of the Kruskal coordinates by the line element

$$ds^2=-\frac{du\,dv}{1-uv} \tag{39}$$

The singularity is at $uv=1$. The region outside of the horizon is given by $u\geq 0\geq v$ or by $v\geq 0\geq u$. The region inside the horizon is given by $u,v\geq 0$ and $1\geq uv\geq 0$. The forbidden region where particle geodesics cannot penetrate classically is $uv>1$. The black hole that emerges from the gauged WZW model SL$(2,R)/R$ is reproduced with the choice $N=\frac{1}{\sqrt{1-uv}\sqrt{2}}$. For lambda follows

$$\Lambda\left(\varepsilon,u,v\right)=\frac{-2\varepsilon_{+'-'}+\left(\varepsilon_{+'+}u+\varepsilon_{+'-}v\right)-\left(\varepsilon_{-'-}v+\varepsilon_{-'+}u\right)\left(-2+uv\right)}{1-uv}. \tag{40}$$

2.2 More Examples:

All $1+1$ black holes are conformally related. In order to find other $1+1$ black holes or two dimensional sections of higher dimensional ones we need only find the right N. Here we would like to show this using Schwarzschild coordinates. Defining $\sqrt{2}u = t + r_*$and $\sqrt{2}v = t - r_*$ the gauge becomes

$$
\begin{aligned}
M &= (+' \qquad -' \qquad 0 \qquad 1) && (41)\\
X^M &= \left(1, \quad \frac{1}{2}\left(r_*^2 - t^2\right), \quad t, \quad r_*\right) N_{(t,r)} && (42)\\
P^M &= \frac{1}{N}(0, \quad r_* p_* - t p_t, \quad p_t, \quad p_*) && (43)\\
(ds)^2 &= N^2[-(dt)^2 + (dr_*)^2] && (44)
\end{aligned}
$$

(or $t = \tau$ and $p_t = |p_*|$ for all three gauge choices). The well known black hole metric

$$ds^2 = -N^2(dt)^2 + N^{-2}(dr)^2 \tag{45}$$

requires $dr_* = drN^{-2}$. Thus we need to integrate to get the right r_* from the N that we want. The right canonical conjugate to r is $p_r = p_* N^{-2}$. Then we can write the gauge in terms of r and p_r via writing the gauge with $r_*(r)$ and $p_*(p_r) = p_r N^2$(For example:$P^{-'} = [r_*(r) p_r N^2 - t p_t]\frac{1}{N}$).

Examples:

Black Hole	N	r_*
BTZ	$\sqrt{\left(\frac{r}{l}\right)^2 - M}$	$\frac{l}{2\sqrt{M}} \ln \frac{r - l\sqrt{M}}{r + l\sqrt{M}}$
Reissner-N.	$\sqrt{1 - \frac{r_+}{r}}\sqrt{1 - \frac{r_-}{r}}$	$r + \frac{r_+^2 \ln(r - r_+) + r_-^2 \ln(r - r_-)}{r_+ - r_-}$

2.3 The Quantum Theory

We quantize via $[u, p_u] = i = [v, p_v]$. It turns out that all operators (like u, p, r, L^{MN}) are already in the ordering that makes them Hermitian relative to the field theoretical dot product. The field theoretical Hermiticity does not imply $L^\dagger = L$ but requires these properties when the L act on fields. The Hermiticity is then $\langle\phi|L\psi\rangle = \langle L\phi|\psi\rangle$ where the dot product is: $\langle\phi|L\psi\rangle = -\frac{i}{2}\int d^{d-1}x[\phi^*(L\psi),_0 - \phi,_0^* L\psi]$.

With the gauges written as above all $L^{MN} = X^M P^N - X^N P^M$ will close correctly. For example the $G_2^{L,R}$, $G_+^{L,R}$ and $G_-^{L,R}$ have the standard $SL\,(2,R)_{L,R}$ commutation rules among themselves

$$\left[G_2^{L,R}, G_\pm^{L,R}\right] = \pm i G_\pm^{L,R}, \quad \left[G_+^{L,R}, G_-^{L,R}\right] = -2i G_2^{L,R}. \tag{46}$$

The conformal factor N does not change the flat space generators in $d = 2$ as shown in [9]. For the field theoretical operators that act on wave functions write $p_u = -i\partial_u$ and so on. The Casimir is $C_2[SO(d,2)] = 0$ as it should be in two dimensions since $C_2 = 1 - \frac{d^2}{4}$ [4]. Physical states are defined via $\langle\phi\,|2p_u p_v|\,\phi\rangle = 0$. This is the $P^M P_M = 0$ constraint due to the Lagrange multiplier A^{22} , here weakly enforced onto the states. This does not break the $SO(d,2)$ symmetry since one can check that for all the generators L^{MN}

$$\langle\delta p_u p_v\rangle = \left\langle \phi \left| \left(L^{MN}\right)^\dagger (p_u p_v) - (p_u p_v) L^{MN} \right| \phi \right\rangle \tag{47}$$

2.3.1 Canonical Gauge

Leaving field theory one could have operators naively Hermitian via the changes $u \to u - \frac{i}{2p_u}$, $v \to v - \frac{i}{2p_v}$ and

$$\begin{aligned}
L^{+'-'} &\to L^{+'-'} - i && (48)\\
L^{-'+} &\to L^{-'+} + iu - \frac{1}{4p_u} = -up_u u - \frac{1}{4p_u} && (49)\\
G_2^{L/R} &\to G_2^{L/R} - \frac{i}{2} && (50)\\
L^{-'-} &\to L^{-'-} + iv - \frac{1}{4p_v} = vp_v v - \frac{1}{4p_v} && (51)
\end{aligned}$$

Then the generator's algebras again close correctly and the Casimir is as desired $C_2 = 0$ also.

3 BTZ Black Hole Gauges

The BTZ black hole [10] is a solution of 2+1 Gravity with negative cosmological constant $\Lambda = -\frac{1}{l^2}$. Its line element can be written

$$ds^2 = -N^2 dt^2 + N^{-2} d\rho^2 + \rho^2 \left(N^\phi\, dt + d\phi\right)^2 \tag{52}$$

where we work with $(8Gl) = 1$ for convenience. Further holds

$$l^2 N^2 = \left(\rho^2 - Ml^2 + \left(\frac{Jl}{2\rho}\right)^2\right) \tag{53}$$

$$N^\phi = -\frac{J}{2\rho^2} \tag{54}$$

and the outer/inner horizon is at

$$\rho_\pm^2 = \frac{1}{2}\left(Ml^2 \pm r_+^2\right) \tag{55}$$

$$r_+^2 = Ml^2\sqrt{1-\left(\frac{J}{Ml}\right)^2} \tag{56}$$

The AdS gauge of [9] can not only be used to describe the global AdS case but also for backgrounds that are locally AdS only. The covering space of AdS space is a BTZ solution with negative mass and the standard method [11] needs analytic continuations, that is it allows imaginary numbers as values for angles and times for instance, in order to yield BTZ black holes from the AdS covering space. This can now be understood in a more natural way as a $Sp(2)$ duality transformation in two time physics [4].

We start out with (124) to (126) of [9]:

$$M = (\,0' \quad , \quad 1' \quad , \quad m) \tag{57}$$

$$X^M = \left(\pm\sqrt{1+X_m^2(x)} \quad ,1 \quad ,X^m(x)\right) \tag{58}$$

$$P^M = \left(\frac{X^m(x)e_m^\mu(x)p_\mu}{\pm\sqrt{1+X_m^2(x)}} \quad ,0 \quad ,e_m^\mu(x)p_\mu\right) \tag{59}$$

where e_m^μ is the inverse of e_μ^m and designed just such that p_μ has the meaning of canonical momentum:

$$e_\mu^m = X,_\mu^m(x) - X^m(x)\frac{X_n(x)X,_\mu^n}{1+X_m^2(x)} \tag{60}$$

Note that $P^2 = 0$ has not been imposed yet, and there still is one more gauge freedom. This we rewrite:

$$M = (\,0' \quad 1' \quad 0 \quad 1 \quad 2\,) \tag{61}$$

$$X^M = (C\gamma, \quad C\sigma, \quad Ss, \quad Sc, \quad 1)\,l \tag{62}$$

where $C^2 - S^2 = \mp 1$, $c^2 - s^2 = \pm 1$ and $\gamma^2 - \sigma^2 = 1$ such that $X^M X_M = 0$. The upper sign will be the solution outside the event horizon and the lower sign will give the inside solution. $dX^M dX_M$ is the line element

$$ds^2 = \left(\mp S^2 \left(\frac{ds}{c}\right)^2 \pm \left(\frac{dC}{S}\right)^2 + C^2 \left(\frac{d\sigma}{\gamma}\right)^2\right) l^2$$

In order to see the BTZ black hole in this reparametrize as follows:

$$c = \cosh \frac{x^+}{l} \Longleftrightarrow s = \sinh \frac{x^+}{l} \tag{63}$$

for the outside solution and

$$c = \sinh \frac{x^+}{l} \Longleftrightarrow s = \cosh \frac{x^+}{l} \tag{64}$$

for the solution inside the event horizon. Further we pick

$$\gamma = \cosh \frac{x^-}{l} \Longleftrightarrow \sigma = \sinh \frac{x^-}{l} \tag{65}$$

It follows

$$ds^2 = \mp S^2 \left(dx^+\right)^2 \pm S^{-2} l^2 (dC)^2 + C^2 \left(dx^-\right)^2 \tag{66}$$

Let us concentrate from now on onwards on the outside solution just to avoid the proliferation of $\pm$. Then C, c, γ are or at least behave like cosh functions in many respects and S, s, σ like sinh ones. Let C, S be only dependent on a parameter R. Using equations (58) and (60) yields the canonical momenta $p_R, p_{\frac{x^\pm}{l}}$ which we will write in the combinations

$$P = \frac{C}{S_{,R}} p_R = \frac{S}{C_{,R}} p_R \tag{67}$$

$$p = \frac{C}{S} p_{\frac{x^+}{l}} \tag{68}$$

$$\pi = \frac{S}{C} p_{\frac{x^-}{l}} \tag{69}$$

Therefore:

$$P^M = \frac{1}{l}\left(\left(S\gamma P + \frac{\sigma}{S}\pi\right), \left(S\sigma P + \frac{\gamma}{S}\pi\right), \left(CsP - \frac{c}{C}p\right), \left(CcP - \frac{s}{C}p\right), 0\right)$$

In lightcone coordinates $X^{\pm} = X^0 \pm X^1$, i.e. $\eta^{+'-'} = -\frac{1}{2} = \eta^{+-}$ we have:

$$\begin{aligned} M &= (\pm' \quad , \quad \pm \quad , \quad 2) & (70)\\ X^M &= l\left(Ce^{\pm\left(\frac{x^-}{l}\right)}, \pm Se^{\pm\left(\frac{x^+}{l}\right)}, 1\right) & (71)\\ P^M &= \frac{1}{l}\left(e^{\pm\frac{x^-}{l}}\left(SP \pm \frac{\pi}{S}\right), \pm e^{\pm\frac{x^+}{l}}\left(CP \mp \frac{p}{C}\right), 0\right) & (72) \end{aligned}$$

Inserting these into the original SO(3,2) action gives

$$\begin{aligned} L &= \dot{X}^M P_M - \frac{A_{22}}{2} P^M P_M \\ &= \frac{\dot{x}^+}{l} p_{\frac{x^+}{l}} + \dot{R} p_R + \frac{\dot{x}^-}{l} p_{\frac{x^-}{l}} \\ &\quad - \frac{A_{22}}{2l^2}\left(-S^{-2} p^2_{\frac{x^+}{l}} + \left(\frac{S}{C_{,R}}\right)^2 p_R^2 + C^{-2} p^2_{\frac{x^-}{l}}\right) & (73) \end{aligned}$$

From the last bracket one can read off the metric

$$\begin{aligned} G^{\mu\nu} &= \frac{1}{l^2} diag\left(-S^{-2}, \left(\frac{S^2}{C^2{}_{,R}}\right), C^{-2}\right) & (74)\\ \Rightarrow G_{\mu\nu} &= l^2 diag\left(-S^2, \left(\frac{C^2{}_{,R}}{S^2}\right), C^2\right) & (75) \end{aligned}$$

which is consistent with the line element (66).

3.1 Classical Generators of the larger Symmetry

The classical generators of the two time Lorentz symmetry according to $L^{MN} = X^M P^N - X^N P^M$ are as follows:

$$\begin{aligned} L^{2M} &= lP^M & (76)\\ L^{0'1'} &= p_{\left(\frac{x^-}{l}\right)} = L^{+'-'} & (77)\\ L^{01} &= p_{\left(\frac{x^+}{l}\right)} = L^{+-} & (78)\\ L^{0'0} &= \gamma s P - \gamma c p - s\sigma\pi & (79)\\ L^{1'1} &= \sigma c P - \sigma s p - c\gamma\pi & (80) \end{aligned}$$

$$
\begin{aligned}
L^{0'1} &= \gamma cP - \gamma sp - c\sigma\pi && (81)\\
L^{1'0} &= \sigma sP - \sigma cp - s\gamma\pi && (82)\\
L^{\pm'+} &= \frac{1}{2}exp\left(+\left(\frac{x^+}{l} \pm \frac{x^-}{l}\right)\right)(p \pm \pi - P) && (83)\\
L^{\mp'-} &= \frac{1}{2}exp\left(-\left(\frac{x^+}{l} \pm \frac{x^-}{l}\right)\right)(p \pm \pi + P) && (84)
\end{aligned}
$$

To split the SO$(2,2)$ subalgebras into $SL(2,R)_L \otimes SL(2,R)_R$ seems not helpful. Given the complexity of the operators, finding the quantum anomalies with $[R, p_R] = i$ etc. would be a tour de force. The $SO(3,2)$ transformations generated by the L^{MN} are again the Poisson brackets $\delta = \frac{1}{2}\epsilon_{MN}\left\{L^{MN},\ \right\}$

$$
\begin{aligned}
\delta R &= (-\epsilon_{0'0}s\gamma - \epsilon_{0'1}c\gamma + \epsilon_{0'2}S\gamma - \epsilon_{1'0}s\sigma \\
&\quad -\epsilon_{1'1}c\sigma + \epsilon_{1'2}S\sigma + \epsilon_{02}Cs + \epsilon_{12}Cc)\frac{C}{S_{,R}} && (85)\\
\delta\left(\frac{x^+}{l}\right) &= (\epsilon_{0'0}c\gamma + \epsilon_{0'1}s\gamma + \epsilon_{1'0}c\sigma + \epsilon_{1'1}s\sigma)\frac{C}{S} \\
&\quad -\epsilon_{01} - \epsilon_{02}\frac{c}{S} - \epsilon_{12}\frac{s}{S} && (86)\\
\delta\left(\frac{x^-}{l}\right) &= (\epsilon_{0'0}s\sigma + \epsilon_{0'1}c\sigma + \epsilon_{1'0}s\gamma + \epsilon_{1'1}c\gamma)\frac{S}{C} \\
&\quad -\epsilon_{0'1'} + \epsilon_{1'2}\frac{\gamma}{C} + \epsilon_{0'2}\frac{\sigma}{C} && (87)
\end{aligned}
$$

With $\dot{X}^\mu$ dependent on R, x^+, x^- the Lagrangian transforms as

$$
\delta L = \delta(\frac{1}{2A_{22}}G_{\mu\nu}\dot{X}^\mu\dot{X}^\nu) \tag{88}
$$

which gives together with the demand that δL should be at most a total derivative how δA_{22} has to look like such that the action is invariant - just like we did before (37). This bigger SO$(3,2)$ symmetry of the particle action on a BTZ background was to our knowledge not known.

3.2 Example I: A Rotating Outside Solution with Arbitrary Mass

Lets actually give examples for the C and S functions. For $C = \cosh R$ follows $S = \sinh R$ and therefore $P = p_R$. Hence the line element to be compared with (66) is

$$ds^2 = -S^2 \left(dx^+\right)^2 + l^2(dR)^2 + C^2 \left(dx^-\right)^2 \tag{89}$$

which holds for the outside of the black hole and is connected to more intuitive coordinates via $x^\pm = (\rho_\pm \frac{t}{l} - \rho_\mp \phi)$. The equations (52) to (55) can now easily be reproduced; i.e. this is a BTZ black hole with $\rho_\pm$ as inner and outer event horizons and the mass and angular momentum given by the arithmetic and harmonic means of $(\frac{\sqrt{2}}{l}\rho_\pm)^2$:

$$\begin{aligned} M &= \frac{1}{l^2}(\rho_+^2 + \rho_-^2) && (90) \\ J &= \frac{2}{l}(\rho_+ \rho_-) && (91) \end{aligned}$$

3.3 Example II: Outside and Inside Solutions for the J=0 Black Hole with Fixed Mass

From the choice $C = \frac{\rho}{l}$ follows $S = \sqrt{\pm(\left(\frac{\rho}{l}\right)^2 - 1)}$ (+ for $\rho \geq l$, $-$ for $l \geq \rho$) and thus $P = \sqrt{\rho^2 - l^2}$. To make contact with well known notation define $t = x^+$, $\phi = \frac{x^-}{l}$ and $\rho = R$, such that (compare (52))

$$\begin{aligned} ds^2 &= -N^2 dt^2 + N^{-2} d\rho^2 + \rho^2 d\phi^2 && (92) \\ N &= \sqrt{\left(\frac{\rho}{l}\right)^2 - 1} && (93) \end{aligned}$$

It is an infinitely leafed space time: $+\infty > \rho \geq 0$ and $-\infty < t, \phi < +\infty$

An identification to $\phi \equiv \phi + 2\pi$ is an identification of points as well in the larger space time having two times.

4 Other 2+1 and n+1 Backgrounds

4.1 Ansatz with Conformal Gauge

The BTZ gauges feature rather involved operators that allow no straight forward generalization to higher dimensions or, since they came from an AdS-gauge, not to other curvatures. The conformal gauge of the 1 + 1 solutions given earlier can easily be generalized (compare (42) :

$$M = (+' \quad -' \quad 0 \quad i) \tag{94}$$

$$X^M = \left(1, \quad \frac{1}{2}\left(r_*^2 - B^2\right), \quad B, \quad r_*\widehat{\Omega}^i\right)\left(\frac{r}{r_*}\right) \tag{95}$$

where $\widehat{\Omega}$ is a unit vector. Demanding the metric $(ds)^2 = -N^2(dt)^2 + N^{-2}(dr)^2 + r^2(d\widehat{\Omega})^2$ needs $\dot{B}\ B' = \dot{r_*}\ r_*'$ in order to cancel the $(dt)(dr)$ term and the conditions

$$(\dot{B}^2 - \dot{r_*}^2)\left(\frac{r}{r_*}\right) = N^2 = \frac{r_*}{r\sqrt{r_*'^2 - B'^2}} \tag{96}$$

($\dot{N}= 0$ is understood). This leads to

$$\partial_r\left(N^2\partial_r B\right) - \partial_t\left(N^{-2}\partial_t B\right) = 0 \tag{97}$$

$$\partial_r\left(N^2\partial_r r_*\right) - \partial_t\left(N^{-2}\partial_t r_*\right) = 0 \tag{98}$$

, i.e. the solutions are scalar fields on the very backgrounds given via N [recall $\nabla^2\Phi = \partial_\mu\left(\sqrt{-G}G^{\mu\nu}\partial_\nu\Phi\right)$]. Even for simple N it leads to non linear PDEs and all one could possibly gain are background metrics that can be put into the conformal gauge form. Two time theory becomes especially rich with $X^M = X^M_{(x^\mu,p^\mu)}$ but we restrict us to the conformal-gauge ansatz because with $X^M = (...)_{(x^\mu)}\, N_{(x^\mu)}$ the metric can be found very easily.

4.2 Near Horizon Gauges

Sleight deviations from the conformal gauge ansatz above led to the following near horizon backgrounds:

4.2.1 Robertson-Bertotti Background in any Dimensions

$$M = (+' \quad -' \quad 0 \quad i)$$

$$X^M = \left(1, \quad \frac{1}{2}\left(r_*^2 - t^2\right), \quad t, \quad r_*\widehat{\Omega}^i\right) N \tag{99}$$

$$P^M = \frac{1}{N}\left(0, \quad r_*p_* - tp_t, \quad p_t, \quad p_*^i\right) \tag{100}$$

$$p_*^i = p_*\left(\widehat{\Omega}^i + \alpha L^{ij}\widehat{\Omega}_j\right) \tag{101}$$

With $r_* = -\left(\frac{M^2}{r}\right)$ and $N = \left(\frac{r}{M}\right)$ follows

$$(ds)^2 = [-N^2(dt)^2 + N^{-2}(dr)^2] + M^2(d\widehat{\Omega})^2 \tag{102}$$

4.2.2 A d=2+1 Near Horizon Gauge

$$\begin{aligned}
M &= (\quad +' \qquad\qquad -' \qquad\qquad 0 \qquad 1 \qquad 2) \\
X^M &= \left(1, \quad \frac{\left(\frac{M^2}{r}\right)^2 - t^2 + (M\phi)^2}{2}, \quad t, \quad -\left(\frac{M^2}{r}\right), \quad M\phi\right) N \qquad (103) \\
P^M &= \frac{1}{N}\left(\quad 0, \quad -\phi p_\phi - \left(\frac{M^2}{r}\right) p_r N^2 - t p_t, \quad p_t, \quad p_r N^2, \quad \frac{p_\phi}{M}\right) (104)
\end{aligned}$$

With M a constant not dependent of r and $N = \left(\frac{r}{M}\right)$ follows $(ds)^2 = [-N^2(dt)^2 + N^{-2}(dr)^2] + r^2(d\phi)^2$. The Lagrangian $L = \dot{X}^M P_M - \frac{1}{2}A^{22}P^M P_M$ is

$$L = -\,\dot{t}\;p_t -\,\dot{r}\;p_r -\,\dot{\phi}\;p_\phi - \frac{A^{22}}{2}\left(-\left(\frac{p_t}{N}\right)^2 + (Np_r)^2 + \left(\frac{p_\phi}{NM}\right)^2\right) \tag{105}$$

Again we would like to stress that the derivation of these systems - here particles in a curved space time - as two time theory gauges proves they have symmetries hidden in the action that have gone unnoticed or whose origin was clouded before and that make them $SO(d,2)$ symmetric systems.

4.3 Backgrounds containing other Black Holes

The following is to demonstrate that basically all black hole backgrounds can be modelled with two time theory gauges. The following is the $\widehat{\Omega}^1 = 0$ subspace of a more complicated gauge.

$$\begin{aligned}
M &= (\quad +' \qquad\qquad -' \qquad\qquad 0 \qquad 1 \qquad i > 1) \\
X^M &= \left(1, \quad \frac{r_*^2 - t^2 + \left(\frac{r}{N}\right)^2}{2}, \quad t, \quad r_*, \quad \frac{r}{N}\widehat{\Omega}^i\right) N \qquad (106) \\
P^M &= \frac{1}{N}(\quad 0, \quad p_r r - t p_t - p_* r_*, \quad p_t, \quad p_*, \quad p_r N\,(\varpi)) \qquad (107)
\end{aligned}$$

Where $(\varpi) = \left(\widehat{\Omega}^i + \alpha L^{ij}\widehat{\Omega}_j\right)$. It follows

$$(ds)^2 = -N^2(dt)^2 + N^{+2}(dr_*)^2 + N^{+2}(d\frac{r}{N})^2 + r^2(d\widehat{\Omega})^2 \qquad (108)$$

such that black hole backgrounds - here non rotating ones - can be gotten via integrating with the desired N :

$$r_* = \int dr \frac{1}{N^2}\sqrt{1-(N-rN')^2} \qquad (109)$$

$N = \left(\frac{r}{M}\right)$ for example gives again $r_* = -\left(\frac{M^2}{r}\right)$. Examples for black holes are:

BTZ:

$$\begin{aligned} N &= \sqrt{\pm\left[\left(\frac{r}{l}\right)^2 - M\right]} \quad \Longrightarrow \quad (N - rN') = \frac{M}{N} &(110)\\ &\Longrightarrow \quad r_* = \int dr \left[\frac{1}{N^2}\sqrt{\pm\left[1-\left(\frac{M}{N}\right)^2\right]}\right] &(111)\end{aligned}$$

$\pm$ are for the inside/outside solutions where for the latter one has to exchange X^0 and X^1 and adjust X^- and P^M in order to still satisfy the $X^M X_M = 0 = P^M P_M$ constraints.

Reissner-Nordstrom:

$$\begin{aligned} N &= (N_+ N_-) \quad ; \quad N_\pm = \sqrt{1-\frac{r_\pm}{r}} &(112)\\ &\Longrightarrow \quad r_* = \pm\int dr \left[\frac{1}{N}\sqrt{\frac{1}{N^2} - \left(2 - \frac{1}{2N_+^2} - \frac{1}{2N_-^2}\right)^2}\right] &(113)\end{aligned}$$

This is the outside solution. The center and inside solutions are again very similar.

5 Conclusions

Many black hole backgrounds have been shown to be two time theory gauges. Thereby we established $Sp(2)$ gauge duality between several backgrounds. We demonstrated that two time theory is certainly rich enough to allow for and unify very many complicated systems. One should bear in mind that all of the above was found with the help of either the AdS gauge or another rather restricted ansatz (namely the conformal gauge) but two time theory is much richer. As well, the $d+2$ gauges are dual to all the other gauges of equal dimensionality that have been published before, as there are the hydrogen atom and the harmonic oscillator [13] and so on.

Particles in black hole backgrounds having a conformal metric and ones near event horizons show conformal symmetries. For $1+1$ and $2+1$ BTZ backgrounds it was explicitly shown how this symmetry which is hidden in the action emerges from the $SO(d,2)$ that acts linearly in the ungauged two time space but is non linearly realized once a gauge surface is fixed. As observed in [12] and [2] already: The $AdS_n \times S^m$ discussion may benefit from this - symmetries hidden in the action can be as vital as the Lorentz boost symmetry for a free particle (A Lagrangian like $L = -m\sqrt{1-\dot{r}^2}$ or the Hamiltonian $H = \sqrt{p^2+m^2}$do not reveal it.).

The method of [11] has been given a natural justification. In order to go from global AdS to the black hole solution one has had to identify (make time periodic) and complexify (use imaginary numbers for real quantities) because both are gauges in a space time with two times. The difference between the gauges is how the gauge slices lie in the bigger space time. The effective exchange of the roles of one of the times with that of one space like direction relative to the gauge surface was formally achieved via the complexification.

The methods used here often remind of purely geometrical embeddings in order to obtain a metric more conveniently. Hence, we would like to stress that all our gauge choices are consistent, dynamical quantum theories that allow for introduction of spin [9] and supersymmetry [14].

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